# Fields.

**Definition:** Let F be a set and let two binary operations called addition (+) and multiplication (·) be defined over the set F. Then the system (F, +, ·) is called a field F if the following conditions are satisfied:

1. Laws of Addition
   - (i) $a+b \in F$, $a, b \in F$ (Closure law)
   - (ii) $a+b = b+a$, $a, b \in F$ (Commutative law)
   - (iii) $a+(b+c) = (a+b)+c$, $a, b, c \in F$ (associative law)
   - (iv) There exists an element 0 in F called zero such that $a+0 = 0+a = a$ for every $a \in F$.
   - (v) For each element $a \in F$, there exists an element $-a$ in F called negative of a such that $a+(-a) = (-a)+a = 0$.

II. Laws of multiplication:
   - (i) $a \cdot b \in F$, $a, b \in F$ (Closure law)
   - (ii) $a \cdot b = b \cdot a$, $a, b \in F$ (Commutative law)
   - (iii) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, $a, b, c \in F$ (associative law)
   - (iv) There exists an element 1 in F called the unit element such that $a \cdot 1 = 1 \cdot a = a$ for every $a \in F$.
   - (v) For each non-zero element a in F, there exists an element $a^{-1}$ in F called the inverse of a such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

III. Distributive law:
   - (i) $a \cdot (b+c) = a \cdot b + a \cdot c$, $a, b, c \in F$
   - (ii) $(b+c) \cdot a = b \cdot a + c \cdot a$, $a, b, c \in F$

**Example:** The set of numbers of the form $a+b\sqrt{2}$ where a and b are rational numbers is a field under addition and multiplication.

Soln:— Let the set be denoted by $S$.

We will first of all show that the set $S$ is an Abelian group w.r.t. addition.

Let $x = a_1 + b_1\sqrt{2}$, $y = a_2 + b_2\sqrt{2}$ and $z = a_3 + b_3\sqrt{2}$.

Then (i) $x + y = (a_1 + a_2) + (b_1 + b_2)\sqrt{2} \in S$

(ii) $x + y = (a_1 + a_2) + (b_1 + b_2)\sqrt{2}$ and $y + x = (a_2 + a_1) + (b_2 + b_1)\sqrt{2} = (a_1 + a_2) + (b_1 + b_2)\sqrt{2}$, for rational numbers are commutative $\therefore x + y = y + x$.

(iii) $x + (y + z) = \{a_1 + (a_2 + a_3)\} + \{b_1 + (b_2 + b_3)\}\sqrt{2}$
$= (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)\sqrt{2}$

Similarly $(x + y) + z = (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)\sqrt{2}$

$\therefore x + (y + z) = (x + y) + z$

(iv) The zero of $S$ is $0 + 0\cdot\sqrt{2} = 0$

(v) The inverse of $a + b\sqrt{2}$ is $-a - b\sqrt{2} \in S$.

Hence $S$ is an additive Abelian group.

Again (i) $x\cdot y = (a_1a_2 + 2b_1b_2) + (a_1b_2 + b_1a_2)\sqrt{2} \in S$

(ii) $y\cdot x = (a_2a_1 + 2b_2b_1) + (a_2b_1 + b_2a_1)\sqrt{2}$
$= (a_1a_2 + 2b_1b_2) + (a_1b_2 + b_1a_2)\sqrt{2}$

$\therefore xy = yx$.

(iii) $x\cdot(y\cdot z) = (a_1 + b_1\sqrt{2})\{(a_2 + b_2\sqrt{2})(a_3 + b_3\sqrt{2})\}$
$= (a_1 + b_1\sqrt{2})\{(a_2a_3 + 2b_2b_3) + (a_2b_3 + b_2a_3)\sqrt{2}\}$.
$= \{a_1a_2a_3 + 2(a_1b_2b_3 + a_2b_3b_1 + a_3b_1b_2)\}$
$+ \sqrt{2}\{2b_1b_2b_3 + (a_2a_3b_1 + a_3a_1b_2 + a_1a_2b_3)\}$

Similarly $(x\cdot y)\cdot z =$ " " " "

$\therefore x\cdot(y\cdot z) = (x\cdot y)\cdot z$.

(iv) The Unity element is $1+0\cdot\sqrt{2} = 1$.

(v) The multiplicative inverse of a non-zero element $a+b\sqrt{2}$ is

$$\frac{1}{a+b\sqrt{2}} = \frac{a-b\sqrt{2}}{(a+b\sqrt{2})(a-b\sqrt{2})} = \frac{a-b\sqrt{2}}{a^2-2b^2}$$

$$= \left(\frac{a}{a^2-2b^2}\right) - \left(\frac{b}{a^2-2b^2}\right)\sqrt{2}$$

Thus the non-zero elements of S form an Abelian group w.r.t. multiplication.

The distributive laws can be satisfied similarly by actual calculation.

Hence the set S is a field.

Theorem:— Every field is an integral domain but the converse is not necessarily true.

Proof:— Since a field F is a commutative ring with Unity, therefore in order to show that every field is an Integral domain, we should show that a field has no zero divisors.

Let $a, b \in F$ with $a \neq 0$ such that $ab = 0$. We shall show that $b = 0$

Since $a \neq 0 \Rightarrow a^{-1}(ab) = a^{-1}(0) = 0$

$\Rightarrow (a^{-1}a)\, b = 0$

$\Rightarrow 1b = 0 \quad \because a^{-1}a = 1$

$\Rightarrow b=0 \quad \because 1b=b$

Similarly, let $b \neq 0$ and $ab=0$

then we have, $ab=0 \Rightarrow (ab)b^{-1} = 0b^{-1}$

$\Rightarrow a(bb^{-1}) = 0$

$\Rightarrow a \cdot 1 = 0 \Rightarrow a=0$

This establishes the fact that if $a, b \in F$, then $ab=0 \Rightarrow a=0$ or $b=0$

Hence a field has no zero divisors.

Therefore every field is an integral domain. But the converse is not true. i.e. every integral domain is not necessarily a field.

Example of an integral domain which is not a field:-

The ring of integers I is a commutative ring with unity. Also I does not possess zero divisors, we know that if $a, b \in I$ such that $ab=0$, then either a or b must be zero.

Hence the ring of integers I is an integral domain but it is not a field since the multiplicative inverse of any non-zero integer $\in I$ does not belong to I.

Anjani Kumar Singh.